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拉瑪奴姜問題 524 的證明

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研究目的

本人試圖尋求定理 1 來證明拉瑪奴姜問題 524, 請看下文的剖析。

研究過程

定理1: 已知 α,β,γ 三實數, $t=-(\alpha+\beta+\gamma)$ 爲一實數, $v=\alpha\beta+\beta\gamma+\gamma\alpha$ 爲一實數, 滿足

(1) $\alpha\beta\gamma=-1$,

(2) $t+v=-3$, 則

$a=\alpha^{\frac{1}{3}}+\beta^{\frac{1}{3}}+\gamma^{\frac{1}{3}}=\sqrt[3]{(-t-6)+3\sqrt[3]{t^2+3t+9}}$。

證明:

一、構造以 α,β,γ 爲三根的一元三次方程式

是 $x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\beta\gamma+\gamma\alpha)x-\alpha\beta\gamma=x^3+tx^2+vx+1=0$。

二、構造以 $\alpha^{\frac{1}{3}},\beta^{\frac{1}{3}},\gamma^{\frac{1}{3}}$ 爲三根的一元三次方程式

1. 設 $a=\alpha^{\frac{1}{3}}+\beta^{\frac{1}{3}}+\gamma^{\frac{1}{3}}$,
2. 設 $b=(\alpha\beta)^{\frac{1}{3}}+(\beta\gamma)^{\frac{1}{3}}+(\gamma\alpha)^{\frac{1}{3}}$,
3. 設 $c=(\alpha\beta\gamma)^{\frac{1}{3}}=(-1)^{\frac{1}{3}}=-1$,
4. 由 1. 2. 3. 得知以 $\alpha^{\frac{1}{3}},\beta^{\frac{1}{3}},\gamma^{\frac{1}{3}}$ 爲三根的一元三次方程式

   是 $z^3-az^2+bz+1=0 \ \Rightarrow\ z^3+1=z(az-b)$。

三、$x=z^3$ 且 $x^3+tx^2+vx+1=0$

$\Rightarrow\ (z^3)^3+t(z^3)^2+v(z^3)+1=0 \ \Rightarrow\ z^9+tz^6+vz^3+1=0$。

四、由二、得知 $z^3+1=z(az-b) \ \Rightarrow\ (z^3+1)^3=z^3(az-b)^3$,

$\Rightarrow\ z^9+3z^6+3z^3+1=z^3\cdot[a^3z^3-3abz(az-b)-b^3]$,

$\Rightarrow\ z^9+3z^6+3z^3+1=z^3\cdot[a^3z^3-3ab(z^3+1)-b^3]$,

$\Rightarrow\ z^9+(3-a^3+3ab)z^6+(3+3ab+b^3)z^3+1=0$。

五、由三、四、得知 $z^9+tz^6+vz^3+1=0$ 與 $z^9+(3-a^3+3ab)z^6+(3+3ab+b^3)z^3+1=0$ 爲同一個方程式

由比較係數可得 $\begin{cases} t=3-a^3+3ab..................(1),\\ v=3+3ab+b^3..................(2).\end{cases}$

(2) 式$-$(1) 式 $\Rightarrow$ $v-t=b^3+a^3$ 且 $v+t=-3$ $\Rightarrow$ $b^3=-a^3-2t-3$. ....(3)

由 (1) 式 $\Rightarrow$ $a^3+t-3=3ab$ $\Rightarrow$ $(a^3+t-3)^3=27a^3b^3$. ....(4)

將 (3) 式代入 (4) 式得

$$\begin{aligned}
&(a^3+t-3)^3=27a^3(-a^3-2t-3)\\
\Rightarrow\ & a^9+3(t-3)a^6+3(t-3)^2a^3+(t-3)^3=-27a^6-54ta^3-81a^3\\
\Rightarrow\ & a^9+3(t+6)a^6+3[t^2-6t+9+18t+27]a^3+(t-3)^3=0\\
\Rightarrow\ & a^9+3(t+6)a^6+3(t^2+12t+36)a^3=-(t-3)^3\\
\Rightarrow\ & a^9+3(t+6)a^6+3(t+6)^2a^3+(t+6)^3=(t+6)^3-(t-3)^3\\
\Rightarrow\ & [a^3+(t+6)]^3=(t^3+18t^2+108t+216)-(t^3-9t^2+27t-27)\\
\Rightarrow\ & [a^3+(t+6)]^3=27t^2+81t+243\\
\Rightarrow\ & [a^3+(t+6)]^3=27(t^2+3t+9),
\end{aligned}$$

因爲 a 與 t 爲實數, 所以 (a^3+t+6) 爲實數

$$\Rightarrow\ a^3+t+6=3\sqrt[3]{t^2+3t+9} \text{ 或 } 3\sqrt[3]{t^2+3t+9}\cdot\omega \text{ (不合) 或 }$$

$$3\sqrt[3]{t^2+3t+9}\cdot\omega^2 \text{ (不合), 其中 } \omega=\frac{-1+\sqrt{3}i}{2}$$

$$\Rightarrow\ a^3=(-t-6)+3\sqrt[3]{t^2+3t+9}$$

$$\Rightarrow\ a=\sqrt[3]{(-t-6)+3\sqrt[3]{t^2+3t+9}},$$

得證定理 1

$$a=\alpha^{\frac{1}{3}}+\beta^{\frac{1}{3}}+\gamma^{\frac{1}{3}}=\sqrt[3]{(-t-6)+3\sqrt[3]{t^2+3t+9}},$$

六、利用定理 1 證明拉瑪奴姜問題 524:

1. 試證: $\sqrt[3]{\cos\dfrac{2\pi}{7}}+\sqrt[3]{\cos\dfrac{4\pi}{7}}+\sqrt[3]{\cos\dfrac{8\pi}{7}}=\sqrt[3]{\dfrac{1}{2}(5-3\sqrt[3]{7})}$。

   證明: 令 $\alpha=-2\cos\dfrac{2\pi}{7}$, $\beta=-2\cos\dfrac{4\pi}{7}$, $\gamma=-2\cos\dfrac{8\pi}{7}$。

$$(1).\quad \alpha+\beta+\gamma=-2\cos\frac{2\pi}{7}-2\cos\frac{4\pi}{7}-2\cos\frac{8\pi}{7}$$

$$= \frac{\sin \frac{2\pi}{7}(-2\cos \frac{2\pi}{7} - 2\cos \frac{4\pi}{7} - 2\cos \frac{8\pi}{7})}{\sin \frac{2\pi}{7}}$$

$$= \frac{-\sin \frac{4\pi}{7} + \sin \frac{2\pi}{7} - \sin \frac{6\pi}{7} + \sin \frac{6\pi}{7} - \sin \frac{10\pi}{7}}{\sin \frac{2\pi}{7}}$$

$$= \frac{-\sin \frac{4\pi}{7} + \sin \frac{2\pi}{7} - \sin \frac{6\pi}{7} + \sin \frac{6\pi}{7} + \sin \frac{4\pi}{7}}{\sin \frac{2\pi}{7}} = 1.$$

(2). $\alpha\beta+\beta\gamma+\gamma\alpha = 4\cos \frac{2\pi}{7} \cdot \cos \frac{4\pi}{7} + 4\cos \frac{4\pi}{7} \cdot \cos \frac{8\pi}{7} + 4\cos \frac{8\pi}{7} \cdot \cos \frac{2\pi}{7}$

$$= 2\left(\cos \frac{2\pi}{7} + \cos \frac{6\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{12\pi}{7} + \cos \frac{6\pi}{7} + \cos \frac{10\pi}{7}\right)$$

$$= 4\left(\cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{6\pi}{7}\right)$$

$$= \frac{4\sin \frac{2\pi}{7}(\cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{6\pi}{7})}{\sin \frac{2\pi}{7}}$$

$$= \frac{2(\sin \frac{4\pi}{7} + \sin \frac{6\pi}{7} - \sin \frac{2\pi}{7} + \sin \frac{8\pi}{7} - \sin \frac{4\pi}{7})}{\sin \frac{2\pi}{7}}$$

$$= \frac{2(\sin \frac{4\pi}{7} + \sin \frac{6\pi}{7} - \sin \frac{2\pi}{7} + \sin \frac{6\pi}{7} - \sin \frac{4\pi}{7})}{\sin \frac{2\pi}{7}} = -2.$$

(3). $\alpha \cdot \beta \cdot \gamma = -8\cos \frac{2\pi}{7} \cdot \cos \frac{4\pi}{7} \cdot \cos \frac{8\pi}{7}$

$$= \frac{\sin \frac{2\pi}{7}(-8\cos \frac{2\pi}{7} \cdot \cos \frac{4\pi}{7} \cdot \cos \frac{8\pi}{7})}{\sin \frac{2\pi}{7}}$$

$$= \frac{-4(\sin \frac{4\pi}{7} \cdot \cos \frac{4\pi}{7} \cdot \cos \frac{8\pi}{7})}{\sin \frac{2\pi}{7}}$$

$$= \frac{-2(\sin \frac{8\pi}{7} \cdot \cos \frac{8\pi}{7})}{\sin \frac{2\pi}{7}} = \frac{-\sin \frac{16\pi}{7}}{\sin \frac{2\pi}{7}} = \frac{-\sin \frac{2\pi}{7}}{\sin \frac{2\pi}{7}} = -1.$$

(4). 由 (1). (2). (3). 得知, 滿足定理1, $\alpha\beta\gamma = -1$ 與 $t + v = -(\alpha + \beta + \gamma) + (\alpha\beta + \beta\gamma + \gamma\alpha) = -(1) + (-2) = -3$,

$$\text{因此}\quad a = \alpha^{\frac{1}{3}} + \beta^{\frac{1}{3}} + \gamma^{\frac{1}{3}} = \sqrt[3]{(-t-6) + 3\sqrt[3]{t^2+3t+9}}$$

$$= \alpha^{\frac{1}{3}} + \beta^{\frac{1}{3}} + \gamma^{\frac{1}{3}} = \sqrt[3]{(1-6) + 3\sqrt[3]{1-3+9}}$$

$$= \alpha^{\frac{1}{3}} + \beta^{\frac{1}{3}} + \gamma^{\frac{1}{3}} = \sqrt[3]{-5+3\sqrt[3]{7}},$$

即 $\sqrt[3]{-2\cos \frac{2\pi}{7}} + \sqrt[3]{-2\cos \frac{4\pi}{7}} + \sqrt[3]{-2\cos \frac{8\pi}{7}} = \sqrt[3]{-5+3\sqrt[3]{7}}$

$$\Rightarrow\ \sqrt[3]{-2}\Big(\sqrt[3]{\cos\frac{2\pi}{7}}+\sqrt[3]{\cos\frac{4\pi}{7}}+\sqrt[3]{\cos\frac{8\pi}{7}}\Big)=\sqrt[3]{-5+3\sqrt[3]{7}}$$

$$\Rightarrow\ \sqrt[3]{\cos\frac{2\pi}{7}}+\sqrt[3]{\cos\frac{4\pi}{7}}+\sqrt[3]{\cos\frac{8\pi}{7}}=\frac{\sqrt[3]{-5+3\sqrt[3]{7}}}{\sqrt[3]{-2}},$$

得證 $\sqrt[3]{\cos\frac{2\pi}{7}}+\sqrt[3]{\cos\frac{4\pi}{7}}+\sqrt[3]{\cos\frac{8\pi}{7}}=\sqrt[3]{\frac{1}{2}(5-3\sqrt[3]{7})}.$

2. 試證: $\sqrt[3]{\cos\frac{2\pi}{9}}+\sqrt[3]{\cos\frac{4\pi}{9}}+\sqrt[3]{\cos\frac{8\pi}{9}}=\sqrt[3]{\frac{3}{2}(\sqrt[3]{9}-2)}.$

證明: 令 $\alpha=2\cos\frac{2\pi}{9}$, $\beta=2\cos\frac{4\pi}{9}$, $\gamma=2\cos\frac{8\pi}{9}$.

$$\begin{aligned}
(1).\quad \alpha+\beta+\gamma &= 2\cos\frac{2\pi}{9}+2\cos\frac{4\pi}{9}+2\cos\frac{8\pi}{9}\\
&= 2\cos(\frac{5\pi}{9}-\frac{3\pi}{9})+2\cos\frac{4\pi}{9}+2\cos(\frac{5\pi}{9}+\frac{3\pi}{9})\\
&= 4\cos\frac{5\pi}{9}\cdot\cos\frac{3\pi}{9}+2\cos\frac{4\pi}{9}=4\cos\frac{5\pi}{9}\cdot(\frac{1}{2})+2\cos\frac{4\pi}{9}\\
&= 2\cos\frac{5\pi}{9}+2\cos\frac{4\pi}{9}=-2\cos\frac{4\pi}{9}+2\cos\frac{4\pi}{9}=0.
\end{aligned}$$

$$\begin{aligned}
(2).\alpha\beta+\beta\gamma+\gamma\alpha &= 4\cos\frac{2\pi}{9}\cdot\cos\frac{4\pi}{9}+4\cos\frac{4\pi}{9}\cdot\cos\frac{8\pi}{9}+4\cos\frac{8\pi}{9}\cdot\cos\frac{2\pi}{9}\\
&= 2\Big(\cos\frac{2\pi}{9}+\cos\frac{6\pi}{9}+\cos\frac{4\pi}{9}+\cos\frac{12\pi}{9}+\cos\frac{6\pi}{9}+\cos\frac{10\pi}{9}\Big)\\
&= 2\Big(\cos\frac{2\pi}{9}+\cos\frac{4\pi}{9}+\cos\frac{10\pi}{9}-\frac{3}{2}\Big)\\
&= 2\Big[\cos\Big(\frac{6\pi}{9}-\frac{4\pi}{9}\Big)+\cos\frac{4\pi}{9}+\cos\Big(\frac{6\pi}{9}+\frac{4\pi}{9}\Big)-\frac{3}{2}\Big]\\
&= 2\Big(2\cos\frac{6\pi}{9}\cdot\cos\frac{4\pi}{9}+\cos\frac{4\pi}{9}-\frac{3}{2}\Big)\\
&= 2\Big(-\cos\frac{4\pi}{9}+\cos\frac{4\pi}{9}-\frac{3}{2}\Big)=-3.
\end{aligned}$$

$$\begin{aligned}
(3).\qquad \alpha\cdot\beta\cdot\gamma &= 8\cos\frac{2\pi}{9}\cdot\cos\frac{4\pi}{9}\cos\frac{8\pi}{9}\\
&= \frac{\sin\frac{2\pi}{9}\Big(8\cos\frac{2\pi}{9}\cdot\cos\frac{4\pi}{9}\cos\frac{8\pi}{9}\Big)}{\sin\frac{2\pi}{9}}\\
&= \frac{4\Big(\sin\frac{4\pi}{9}\cdot\cos\frac{4\pi}{9}\cdot\cos\frac{8\pi}{9}\Big)}{\sin\frac{2\pi}{9}}\\
&= \frac{2\Big(\sin\frac{8\pi}{9}\cdot\cos\frac{8\pi}{9}\Big)}{\sin\frac{2\pi}{9}}=\frac{\sin\frac{16\pi}{9}}{\sin\frac{2\pi}{9}}=\frac{-\sin\frac{2\pi}{9}}{\sin\frac{2\pi}{9}}=-1.
\end{aligned}$$

(4). 由 (1). (2). (3). 得知, 滿足定理 1, $\alpha\beta\gamma=-1$ 與 $t+v=-(\alpha+\beta+\gamma)+(\alpha\beta+\beta\gamma+\gamma\alpha)=-(0)+(-3)=-3$

$$\text{因此}\quad a=\alpha^{\frac{1}{3}}+\beta^{\frac{1}{3}}+\gamma^{\frac{1}{3}}=\sqrt[3]{(-t-6)+3\sqrt[3]{t^2+3t+9}}$$
$$=\alpha^{\frac{1}{3}}+\beta^{\frac{1}{3}}+\gamma^{\frac{1}{3}}=\sqrt[3]{(-0-6)+3\sqrt[3]{0+0+9}}$$
$$=\alpha^{\frac{1}{3}}+\beta^{\frac{1}{3}}+\gamma^{\frac{1}{3}}=\sqrt[3]{3\sqrt[3]{9}-6},$$
$$\text{即}\ \sqrt[3]{2\cos\frac{2\pi}{9}}+\sqrt[3]{2\cos\frac{4\pi}{9}}+\sqrt[3]{2\cos\frac{8\pi}{9}}=\sqrt[3]{3\sqrt[3]{9}-6},$$
$$\Rightarrow\ \sqrt[3]{2}\cdot\left(\sqrt[3]{\cos\frac{2\pi}{9}}+\sqrt[3]{\cos\frac{4\pi}{9}}+\sqrt[3]{\cos\frac{8\pi}{9}}\right)=\sqrt[3]{3\sqrt[3]{9}-6}$$
$$\Rightarrow\ \sqrt[3]{\cos\frac{2\pi}{9}}+\sqrt[3]{\cos\frac{4\pi}{9}}+\sqrt[3]{\cos\frac{8\pi}{9}}=\frac{\sqrt[3]{3\sqrt[3]{9}-6}}{\sqrt[3]{2}}$$
$$\text{得證}\ \sqrt[3]{\cos\frac{2\pi}{9}}+\sqrt[3]{\cos\frac{4\pi}{9}}+\sqrt[3]{\cos\frac{8\pi}{9}}=\sqrt[3]{\frac{3}{2}(\sqrt[3]{9}-2)}.$$

參考文獻

1. S. Ramanujan, Ramanujan question 524, Submitted by *Ramanujan to the Journal of the Indian Mathematical Society.*

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